JOURNAL OF APPROXIMATION THEORY 6, 80–89 (1972)

Concerning Weighted Approximation, Vector Fibrations, and Algebras of Operators

LEOPOLDO NACHBIN AND JOÃO B. PROLLA

University of Rochester, Rochester, New York 14627

AND

SILVIO MACHADO

University of Wisconsin, Madison, Wisconsin 53706

Communicated by Oved Shisha

Received October 27, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

In this article we present a different approach in proving the results contained in our previous paper [6]. These results were concerned with weighted locally convex spaces of cross sections and with algebras of operators. (See Section 2 for definitions.) The viewpoint we shall adopt here consists in firstly proving the so-called bounded case of the weighted approximation problem, and then use it to treat the general case. This approach corresponds to the one used in [4] for the case of modules of continuous functions, whereas the approach of [6] corresponds to the one used in [5].

The weighted spaces of cross sections contain as a particular case the weighted spaces of vector-valued functions. For these it is possible to generalize many of the results about scalar-valued functions which do not generalize to cross sections. For such generalizations see [8], where the weighted Dieudonné theorem for density in tensor products is treated; [9], where the dual of a weighted space of continuous vector-valued functions on a locally compact space is determined; and [10], which concerns the non-self-adjoint bounded case of the weighted approximation problem.

2. Weighted Locally Convex Spaces of Cross-Sections

A *vector fibration* is a pair $(E, (F_x)_{x\in E})$ where E is a Hausdorff space and $(F_x)_{x\in E}$ is a family of vector spaces, each vector space over the same field $\mathbf{K}$ of scalars ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). By a *cross section over E* we mean any element of the Cartesian product $\prod_{x\in E} F_x$, i.e., any function f defined on E, and such that $f(x) \in F_x$ for all $x \in E$. The Cartesian product $\prod_{x\in E} F_x$ is made a vector space in the usual way, and a vector space of cross sections over E is, by definition, any vector subspace of $\prod_{x\in E} F_x$.

A *weight on E* is a function v defined on E and such that $v(x)$ is a seminorm over F_x for each $x \in E$. A set V of weights on E is said to be *directed* if, for every pair $v_1, v_2 \in V$, there exist $v \in V$ and $t > 0$ such that $v_i(x) \leqslant tv(x)$ for all $x \in E$, $i = 1, 2$. From now on we shall consider only directed sets of weights.

If f is a cross section over E and v is a weight on E, we will denote by $v[f]$ the positive valued function defined on E by $x \mapsto v(x)[f(x)]$.

If X is a subset of E, then $(X, \prod_{x\in X} F_x)$ is a vector fibration, and for any cross section f over E, its restriction $f \mid X$ is a cross section over X. Similarly, if v is a weight on E, its restriction $v \mid X$ is a weight on X, and obviously $v \mid X[f \mid X] = v[f] \mid X$. If L is a vector space of cross sections over E, we will denote by $L \mid X$ the vector space of all $f \mid X$ where f ranges over L. Obviously $L \mid X$ is a vector space of cross sections over X. Similarly, we denote by $V \mid X$ the set of all restrictions $v \mid X$ where v ranges over V.

Definition 1. Let L be vector space of cross sections over E. A weight v on E is said to be

(1) L-bounded,

(2) L-upper semicontinuous,

(3) L-null at infinity,

in case the function $v[f]$ is, respectively,

(1) bounded on E,

(2) upper semicontinuous on E,

(3) null at infinity on E, for every cross section $f \in L$.

From this definition it follows that any weight v which is L-bounded determines a seminorm over L, namely,

$$f \mapsto \|f\|_v = \sup\{v(x)[f(x)];\ x \in E\}.$$

Notice also that if a weight v is L-upper semicontinuous and L-null at infinity, then v is L-bounded.

DEFINITION 2. Let L be a vector space of cross sections over E, and let V be a directed set of weights which are L-bounded. We will denote by LV_b the locally convex space obtained by endowing L with the topology determined by the family of seminorms $f \mapsto \|f\|_v$, where v ranges over V. If the weights $v \in V$ are L-upper semicontinuous and L-null at infinity, LV_∞ will denote the locally convex space obtained as above. The spaces LV_b and LV_∞ are called weighted locally convex spaces of cross sections.

Since we assumed V to be directed, the sets of the form $\{f \in L; \|f\|_v \leqslant \epsilon\}$, where $v \in V$ and $\epsilon > 0$, form a basis of neighborhoods of the origin in LV_b or LV_∞.

When X is a closed subset of E and v is an L-upper semicontinuous weight on E, then $v \mid X$ is $(L \mid X)$-upper semicontinuous. Similar properties hold for weights that are L-bounded or L-null at infinity. Hence if LV_b or LV_∞ are defined, then $(L \mid X)(V \mid X)_b$ or $(L \mid X)(V \mid X)_\infty$ are also defined. We will denote such spaces simply by $LV_b \mid X$ and $LV_\infty \mid X$, respectively. For more details, see [1, 6].

3. THE WEIGHTED APPROXIMATION PROBLEM

The vector space $\prod_{x \in E} F_x$ of all cross sections is an *A-module*, for any subalgebra $A \subset \mathscr{C}(E; \mathbf{K})$, under the following multiplication operation: if $u \in A$ and f is a cross section, then uf is the cross section whose value at $x \in E$ is $u(x) f(x)$. If W is a vector space of cross sections, we say that W is an A-module if W is an A-submodule of $\prod_{x \in E} F_x$.

Given an A-module $W \subset LV_\infty$, the *weighted approximation problem* consists, then, in asking for a description of the closure of W in LV_∞; and, in particular, in finding necessary and sufficient conditions for W to be dense in LV_∞.

In the special case in which A consists only of constant functions, an A-module is, in general, only a vector subspace of LV_∞. In such a case, the only thing we can do is the following: Once the dual of LV_∞ is known, apply the Hahn–Banach theorem.

We shall try to reduce the general case to this special one by looking at the subsets of E on which the functions of A are constant, namely, the equivalence classes $X \subset E$ modulo the equivalence relation: $x_1 \sim x_2$, whenever $x_1, x_2 \in E$ and $u(x_1) = u(x_2)$ for all $u \in A$. We shall denote this equivalence relation by E/A.

DEFINITION 3. An A-module $W \subset LV_\infty$ is said to be localizable under A in LV_∞ if its closure in LV_∞ consists of those $f \in L$ for which $f \mid X$ belongs to the closure of $W \mid X$ in $LV_\infty \mid X$ for each equivalence class $X \subset E$ modulo E/A.

The *strict weighted approximation problem* consists, then, in asking for necessary and sufficient conditions in order that W be localizable under A in LV_∞.

Suppose that $A \subset \mathscr{C}(E; \mathbf{K})$ is *separating on* E, that is : if $x, y \in E$, $x \neq y$, there exists $a \in A$ such that $a(x) \neq a(y)$. Let $W \subset LV_\infty$ be an A-module which is localizable under A in LV_∞. It follows from the above definitions that in this case W is dense in LV_∞ if and only if, for each $x \in E$, $W(x) = \{w(x); w \in W\}$ is dense in $L(x) = \{f(x); f \in L\} \subset F_x$, where F_x is endowed with the topology determined by the family of seminorms $V(x) = \{v(x); v \in V\}$.

4. The Separating Case

Let LV_∞ be a weighted locally convex space of crosssections and $W \subset LV_\infty$ an A-module. Let F be the quotient space of E by the equivalence relation E/A and let $\pi_* : \mathscr{C}(F; \mathbf{K}) \to \mathscr{C}(E; \mathbf{K})$ be the induced homomorphism defined by $\pi_*(b) = b \cdot \pi$ for all $b \in \mathscr{C}(F; \mathbf{K})$. Then $B = \pi_*^{-1}(A)$ is a subalgebra of $\mathscr{C}(F; \mathbf{K})$ which is separating on F. Hence F is a Hausforff space. For every $y \in F$, $\pi^{-1}(y)$ is a closed subset of E. Let $(F, (G_y)_{y \in F})$ be the vector fibration obtained by defining $G_y = L \mid \pi^{-1}(y)$. For every weight $v \in V$, we define a a corresponding weight u on F, by setting

$$(*) \qquad u(y)[f \mid \pi^{-1}(y)] = \sup\{v(x)[f(x)]; x \in \pi^{-1}(y)\}.$$

Let $M \subset \prod_{y \in F} G_y$ be the vector subspace of cross sections over F, given by $\{(f \mid \pi^{-1}(y)); f \in L\}$, and let U be the set of weights u defined by (*) where v ranges over V. Then each weight $u \in U$ is M-upper semicontinuous and M-null at infinity. This fact results from the following

Lemma (*Lemma* 1 [6]). *Let E and F be two Hausdorff spaces and $\pi : E \to F$ a continuous mapping from E onto F. For any upper semicontinuous function $g : E \to \mathbf{R}_+$ that vanishes at infinity, let $h : F \to \mathbf{R}_+$ be defined by*

$$h(y) = \sup\{ g(x); x \in \pi^{-1}(y)\}$$

for all $y \in F$. Then h is upper semicontinuous and vanishes at infinity on F.

Hence we may consider the weighted space MU_∞. If we define $X = \{(w \mid \pi^{-1}(y)); w \in W\}$, then $X \subset MU_\infty$ and is a B-module.

Theorem 1. *W is localizable under A in LV_∞ if and only if, X is localizable under B in MU_∞.*

Remark 1. Theorem 1 confirms the conjecture stated in [3], namely, that the separating and the general cases of the strict weighted approximation problem are equivalent. This together with the final comments in Section 3 establish that corresponding to every sufficient condition for localizability there is a corollary on density in the separating case.

The argument used to prove Theorem 1 of [6] applies here, too, with only a slight modification.

5. The Bounded Case

From now on, E denotes a completely regular Hausdorff space.

Definition 4. In the notation of Definition 3, the bounded case of the weighted approximation problem occurs when every $a \in A$ is bounded on the support of every $v \in V$. Each of the following hypotheses leads to an instance of the bounded case:

$$A \subset \mathscr{C}_b(E; \mathbf{K}); \tag{1}$$

$$\text{each } v \in V \text{ has a compact support.} \tag{2}$$

Theorem 2. *Assume that A is self-adjoint, in the complex case, and that we are in the bounded case. Then W is localizable under A in LV_∞.*

Proof. Let $f \in LV_\infty$ be such that $f \mid X$ belongs to the closure of $W \mid X$ in $LV_\infty \mid X$, for each equivalence class $X \subset E$ modulo E/A. Let $v \in V$ and $\epsilon > 0$ be given. We may assume $A \subset \mathscr{C}_b(E; \mathbf{K})$ by replacing E by the support of v, if necessary. Given any equivalence class $X \subset E$ modulo E/A, there exists some $w_X \in W$ such that

$$v(x)[f(x) - w_X(x)] < \epsilon$$

for any $x \in X$. The closed set $K_X = \{x \in E; v(x)[f(x) - w_X(x)] \geqslant \epsilon\}$ is compact, since $v[f - w_X]$ vanishes at infinity. Moreover, X and K_X are disjoint. By Lemma 1 [4], there is a finite set $\mathscr{L}$ of equivalence classes in E modulo E/A, and functions φ_X belonging to the closure of A in $\mathscr{C}_b(E; \mathbf{K})$ such that $\varphi_X \geqslant 0$ and $\varphi_X \mid K_X = 0$ for all $X \in \mathscr{L}$ and $\sum_{X \in \mathscr{L}} \varphi_X = 1$. Notice that

$$\varphi_X(x)\, v(x)[f(x) - w_X(x)] \leqslant \epsilon \varphi_X(x) \tag{3}$$

for any $x \in E$ and $X \in \mathscr{L}$. In fact, either $x \in K_X$ and then $\varphi_X(x) = 0$; or else $x \notin K_X$, in which case $v(x)[f(x) - w_X(x)] < \epsilon$. In both cases, (3) holds true. Hence

$$v(x) \left[\sum_{X \in \mathscr{L}} \varphi_X(x)\, w_X(x) - f(x)\right] \leqslant \epsilon, \tag{4}$$

for any $x \in E$. If $\mathscr{L}$ has k elements, let $\delta > 0$ be such that $\delta k M \leqslant \epsilon$, where M is the maximum of $\| w_X \|_v$ for X ranging over $\mathscr{L}$. For each $X \in \mathscr{L}$ there exists some $a_X \in A$ such that $| a_X(x) - \varphi_X(x)| \leqslant \delta$ for all $x \in E$. Hence

$$v(x) \left[\sum_{X \in \mathscr{L}} a_X(x)\, w_X(x) - f(x) \right] \leqslant 2\epsilon$$

for all $x \in E$. Since $AW \subset W$, $w = \sum_{X \in \mathscr{L}} a_X w_X$ belongs to W, and therefore f belongs to the closure of W in LV_∞. Q.E.D.

6. Sufficient Conditions for Localizability

We will denote by $\mathscr{P}(\mathbf{R}^n)$ the algebra of all $\mathbf{R}$-valued polynomials on $\mathbf{R}^n$. A *weight on* $\mathbf{R}^n$ is an upper semicontinuous positive real-valued function on $\mathbf{R}^n$. A weight ω on $\mathbf{R}^n$ is said to be *rapidly decreasing at infinity* when $\mathscr{P}(\mathbf{R}^n) \subset \mathscr{C}\omega_b(\mathbf{R}^n)$, or equivalently $\mathscr{P}(\mathbf{R}^n) \subset \mathscr{C}\omega_\infty(\mathbf{R}^n)$. If, in addition to this, $\mathscr{P}(\mathbf{R}^n)$ is dense in $\mathscr{C}\omega_\infty(\mathbf{R}^n)$, then ω is said to be a *fundamental weight*. We shall denote by Ω_n the set of all fundamental weights on $\mathbf{R}^n$, and by Γ_n the subset of Ω_n consisting of all $\gamma \in \Omega_n$ such that $\gamma^k \in \Omega_n$, for all $k > 0$.

We shall consider $\mathbf{R}^n$ as a vector lattice in the usual way: if $u = (u_1, \ldots, u_n)$ and $t = (t_1, \ldots, t_n)$ belong to $\mathbf{R}^n$, we write $u \leqslant t$ provided $u_i \leqslant t_i$ for all $i = 1, 2, \ldots, n$; and define $| u | = (| u_1 |, \ldots, | u_n |)$. A real-valued function φ defined on $\mathbf{R}^n$ is then said to be *modulus-decreasing* if $u, t \in \mathbf{R}^n$ and $| u | \leqslant | t |$ imply $\varphi(u) \geqslant \varphi(t)$. Denote by $\Omega_n{}^d$ the subset of Ω_n consisting of those fundamental weights which are decreasing, and by $\Gamma_n{}^d$ the intersection $\Gamma_n \cap \Omega_n{}^d$.

If A is a subalgebra of $\mathscr{C}(E; \mathbf{K})$ containing the constants, $G(A)$ will denote a subset of A which topologically generates A as an algebra over $\mathbf{K}$ with unity, i.e., the subalgebra over $\mathbf{K}$ of A, generated by $G(A)$ and 1, is dense in A for the compact-open topology of $\mathscr{C}(E; \mathbf{K})$. Similarly, if $W \subset LV_\infty$ is an A-module, $G(W)$ will denote a subset of W which topologically generates W as a module over A, i.e., the submodule over A of W, generated by $G(W)$, is dense in W for the topology of LV_∞.

Theorem 3. *Suppose that there exist $G(A)$ and $G(W)$ such that*

(1) *$G(A)$ consists of real-valued functions;*

(2) *given any $v \in V$; $a_1, \ldots, a_n \in G(A)$ and $w \in G(W)$, there exist $a_{n+1}, \ldots, a_N \in G(A)$, where $N \geqslant n$, and $\omega \in \Omega_N$, such that*

$$v(x)[w(x)] \leqslant \omega(a_1(x), \ldots, a_n(x), \ldots, a_N(x))$$

for all $x \in E$.

Then W is localizable under A in LV_∞.

Remark 2. The above theorem reduces the search for sufficient conditions for localizability on a completely regular space to the search of sufficient conditions for a weight on $\mathbf{R}^n$ to be fundamental. Theorem 3 follows from Theorem 2 in the same manner as Theorem 2 follows from Theorem 1 [4]. An independent proof of Theorem 3 can be modeled on the proof of Theorem 1 [5, § 26], an approach that was indicated in [6]. Our next theorem is a slight variation of Theorem 3, dropping the hypothesis (1) in the complex case.

THEOREM 4. *Suppose that A is self-adjoint in the complex case, and that there exist $G(A)$ and $G(W)$ such that, given any $v \in V$, $a_1, \ldots, a_n \in G(A)$ and $w \in G(W)$, there exist $a_{n+1}, \ldots, a_N \in G(A)$, where $N \geqslant n$ and $\omega \in \Omega_N{}^d$, such that*

$$v(x)[w(x)] \leqslant \omega(|a_1(x)|, \ldots, |a_n(x)|, \ldots, |a_N(x)|)$$

for all $x \in E$. Then W is localizable under A in LV_∞.

Our next two theorems reduce the search for sufficient conditions for localizability of modules to the search for fundamental weights on $\mathbf{R}$, i.e., to the one-dimensional Bernstein approximation problem.

THEOREM 5. *Suppose that there exist $G(A)$ and $G(W)$ such that*

(1) *$G(A)$ consists of real-valued functions;*

(2) *given any $v \in V$, $a \in G(A)$ and $w \in G(W)$, there exists $\gamma \in \Gamma_1$ such that, for all $x \in E$:*

$$v(x)[w(x)] \leqslant \gamma(a(x)).$$

Then W is localizable under A in LV_∞.

THEOREM 6. *Assume that A is self-adjoint in the complex case, and that there exist $G(A)$ and $G(W)$ such that, given any $v \in V$, $a \in G(A)$ and $w \in G(W)$, there exists $\gamma \in \Gamma_1{}^d$ such that*

$$v(x)[w(x)] \leqslant \gamma(|a(x)|)$$

for all $x \in E$. Then W is localizable under A in LV_∞.

Remark 3. The above theorem combined with classical results concerning the Bernstein problem allows one to find practical sufficient conditions for localizability.

THEOREM 7 (*analytic criterion for localizability*). *Assume that A is self-adjoint in the complex case, and that there exist $G(A)$ and $G(W)$ such that, given* *any* $v \in V$, $a \in G(A)$ *and* $w \in G(W)$, *there exist constants* $C > 0$ *and* $c > 0$ *such that, for all* $x \in E$:

$$v(x)[w(x)] \leqslant Ce^{-c|a(x)|}.$$

Then W *is localizable under* A *in* LV_∞ .

THEOREM 8 (quasi-analytic criterion for localizability). *Assume that* A *is self-adjoint in the complex case, and that there exist* $G(A)$ *and* $G(W)$ *such that, given any* $v \in V$, $a \in G(A)$ *and* $w \in G(W)$, *we have*

$$\sum_{m=1}^{\infty} (M_m)^{-1/m} = +\infty$$

where $M_m = \sup\{v(x)[a^m(x)\, w(x)];\, x \in E\}$ *for* $m = 0, 1, 2,\ldots.$ *Then* W *is localizable under* A *in* LV_∞ .

Remark 4. Theorem 7 is based on the uniqueness of analytic continuation, whereas Theorem 8 rests on the Denjoy–Carleman theorem.

If there exist $G(A)$ and $G(W)$ such that every a $\in G(A)$ is bounded on the support of the function $v[w]$, for any $v \in V$ and $w \in G(W)$, it follows from Theorem 7 that W is localizable under A in LV_∞. This result extends Theorem 2.

7. ALGEBRAS OF OPERATORS

In what follows, $\mathscr{L}$ denotes a locally convex Hausdorff space over $\mathbf{K}$, and $\mathfrak{A}$ denotes a commutative algebra of linear operators over $\mathscr{L}$, not necessarily continuous. We further assume that $\mathfrak{A}$ contains the identity operator.

DEFINITION 5. The point co-spectrum of $\mathfrak{A}$ is the set of all homomorphisms h of $\mathfrak{A}$ onto $\mathbf{K}$ for which there exists $\varphi \in \mathscr{L}'$, $\varphi \neq 0$, such that $\varphi(u(x)) = h(u)\, \varphi(x)$ for all $u \in \mathfrak{A}$, and $x \in \mathscr{L}$.

The point cospectrum of $\mathfrak{A}$ is also the set of all homomorphisms h of $\mathfrak{A}$ onto $\mathbf{K}$ for which there exists $\varphi \in \mathscr{L}'$, $\varphi \neq 0$, such that $\varphi(u(x)) = 0$ for all u in the kernel of h, and x in $\mathscr{L}$. Or, equivalently, the set of all homomorphisms h of $\mathfrak{A}$ onto $\mathbf{K}$ such that the closed vector subspace S_h of $\mathscr{L}$ spanned by $\{u(x);\, u \in h^{-1}(0),\, x \in \mathscr{L}\}$ is a proper vector subspace of $\mathscr{L}$.

We shall endow the point cospectrum of $\mathfrak{A}$ with the weakest topology under which all the functions $\tilde{u}$ defined on it by $\tilde{u}(h) = h(u)$ are continuous, where u ranges over $\mathfrak{A}$. This topology is a Hausdorff one, and we shall denote by E the point cospectrum of $\mathfrak{A}$ endowed with this topology. For each $h \in E$, consider the quotient vector space $F_h = \mathscr{L}/S_h$, and let $x \mapsto x_h$ be the associated quotient map. Then, for each $x \in \mathscr{L}$, the family $(x_h)_{h \in E}$ is a cross-section over E, which we shall denote by $\Phi(x)$. The mapping Φ from $\mathscr{L}$ into $\prod_{h \in E} F_h$ is obviously linear. Let $L = \Phi(\mathscr{L})$. For each continuous seminorm p over $\mathscr{L}$, let p_h denote the quotient seminorm defined by

$$p_h(x_h) = \inf\{p(y);\, y \in x_h\}$$

for all $x_h \in F_h$. The mapping $h \mapsto p_h$ is then a weight over E, and we will denote by $V(\Gamma)$ the set of all such weights, where p ranges over a set Γ of continuous seminorms of $\mathscr{L}$ which determine the topology of $\mathscr{L}$. Notice that every weight in $V(\Gamma)$ is L-bounded, for $p_h(x_h) \leqslant p(x)$ for all $h \in E$. Hence we may consider the weighted space $LV(\Gamma)_b$.

The above inequality also shows that Φ is a continuous map from $\mathscr{L}$ onto $LV(\Gamma)_b$. On the other hand, the mapping $u \mapsto \tilde{u}$ is a homomorphism of $\mathfrak{A}$ into $\mathscr{C}(E; \mathbf{K})$. Let A denote the image of $\mathfrak{A}$ under this homomorphism. Notice that A is separating over E, and that $\Phi(u(x)) = \tilde{u} \cdot \Phi(x)$ for all $u \in \mathfrak{A}$ and $x \in \mathscr{L}$. Hence L is an A-module, and $u \mapsto \tilde{u}$ is an isomorphism whenever Φ is an isomorphism. The following representation theorem establishes a condition under which Φ is a topological vector isomorphism.

THEOREM 9. *A necessary and sufficient condition for the existence of a set Γ of seminorms over $\mathscr{L}$, which determines the topology of $\mathscr{L}$, such that Φ is a topological vector isomorphism between $\mathscr{L}$ and $LV(\Gamma)_b$ is that $\mathscr{L}$ be locally convex under $\mathfrak{A}$ with respect to the category of all algebras isomorphic to* $\mathbf{K}$.

Remark 5. The above notion of local convexity was introduced in [2]. In order to be able to represent $\mathscr{L}$ as an $LV(\Gamma)_\infty$ space, additional hypotheses on the seminorms of Γ must be satisfied namely, for each $p \in \Gamma$ the function $h \to p_h(x_h)$ must be upper semicontinuous and null at infinity, for every $x \in \mathscr{L}$. Once L has been represented as an $LV(\Gamma)_\infty$, we may define localizability under $\mathfrak{A}$ for $\mathfrak{A}$-invariant subspaces, and consider the problem of finding necessary and sufficient conditions for a given $\mathfrak{A}$-invariant subspace to be dense in $\mathscr{L}$. Furthermore, we may ask when spectral synthesis holds, i.e., when a proper closed $\mathfrak{A}$-invariant subspace is the intersection of all the proper closed $\mathfrak{A}$-invariant subspaces of codimension one containing it. The following theorem answers this question (see [6], [7].)

THEOREM 10. *Let $\mathscr{L}$ be a space which can be represented as an $LV(\Gamma)_\infty$, and let $\mathscr{W}$ be a proper closed $\mathfrak{A}$-invariant subspace which is localizable under $\mathfrak{A}$ in $\mathscr{L}$. Then $\mathscr{W}$ is contained in some proper closed $\mathfrak{A}$-invariant subspace of codimension one and is the intersection of all proper closed $\mathfrak{A}$-invariant subspaces of codimension one which contain it.*

References

1. S. Machado, Aproximação ponderada em fibrados vetoriais, *An. Acad. Brasil. Ci.* **43** (1971), 1–21.
2. L. Nachbin, Algebras of finite differential order and the operational calculus, *Ann. of Math.* **70** (1959), 413–437.
3. L. Nachbin, On the weighted polynomial approximation in a locally compact space, *Proc. Nat. Acad. Sci. U. S. A.* **47** (1961), 1055–1057.
4. L. Nachbin, Weighted approximation for algebras and modules of continuous functions: Real and self-adjoint cases, *Ann. of Math.* **81** (1965), 289–302.
5. L. Nachbin, "Elements of Approximation Theory," Van Nostrand, Princeton, NJ, 1967.
6. L. Nachbin, S. Machado and J. B. Prolla, Weighted approximation, vector fibrations and algebras of operators, *J. Math. Pures Appl.* **50** (1971), 299–323.
7. J. B. Prolla, Aproximação ponderada e álgebras de operadores, *An. Acad. Brasil. Ci.* **43** (1971), 23–36.
8. J. B. Prolla, The weighted Dieudonné theorem for density in tensor products, *Indag. Math.* **33** (1971), 170–175.
9. J. B. Prolla, Weighted spaces of vector-valued continuous functions, *Ann. Mat. Pura Appl.* **89** (1971), 145–158.
10. J. B. Prolla, Bishop's generalized Stone-Weierstrass theorem for weighted spaces, *Math. Ann.* **191** (1971), 283–289.